Connections between the soliton dynamics provided by some integrable relativistic theories

L. Martínez Alonso and E. Olmedilla Moreno
Departamento de Métodos Matemáticos de la Física, Facultad de Ciencias Físicas, Universidad de Madrid, 28040 Madrid, Spain

(Received 3 December 1984)

The soliton dynamics in the complex sine-Gordon equation, massive Thirring model, and sine-Gordon equation are shown to be closely related.

The existence of soliton solutions is one of the main features of nonlinear wave models solvable by means of the inverse scattering method. Each of these models has associated a particular type of soliton; however, solitons always share a series of common properties: they behave as classical particles and their scattering processes may be interpreted as a succession of paired collisions in which every soliton collides with all others. In this way, the soliton dynamics provided by an integrable model can be formulated as a classical pure S-matrix theory. There are several relativistically invariant field theories in two-dimensional space-time which can be solved through the inverse scattering method; therefore, it seems natural to investigate the correspondences among these relativistic models by considering their associated soliton dynamics. We have recently analyzed[1] the relationship between the massive Thirring (MT) and sine-Gordon (SG) models whose Lagrangian densities are

$$L=\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi-\tfrac{1}{2}g^{2}(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi) \tag{1}$$

and

$$L=\tfrac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi+\frac{m'^{4}}{g'}\cos\left(\frac{g'^{1/2}}{m'}\phi\right), \tag{2}$$

respectively. We found that the soliton dynamics of the SG equation can be reproduced in terms of the MT model provided the parameters in (1) and (2) are related according to

$$m'=m,\quad g'=4m^{2}g^{2}. \tag{3}$$

In the present paper we show that from the point of view of soliton dynamics the complex sine-Gordon (CSG) model[2,3]

$$L=\frac{1}{2}\left(\frac{|\psi_{\mu}|^{2}}{1-g^{2}|\psi|^{2}}-m^{2}|\psi|^{2}\right) \tag{4}$$

is strongly related to (1) and (2); here ψ denotes a complex scalar field in two-dimensional space-time. The parameters m and g in (4) will be assumed to coincide with those of the MT model (1).

The soliton of the CSG model has the form[3]

$$\psi(t,x)=\frac{1}{g}\sin\theta\,\frac{\exp\{i[m\gamma\cos\theta(t-vx)+\phi]\}}{\cosh[m\gamma\sin\theta(x-q-vt)]}, \tag{5}$$
$$\gamma=(1-v^{2})^{-1/2}.$$

It is characterized by four parameters (q,v,θ,ϕ) such that

$$q,v\in R,\quad 0<\theta<\pi,\quad \phi\in R\ (\mathrm{mod}\,2\pi). \tag{6}$$

The same parameters with identical ranges of variation appear in the expression of the soliton associated with the MT model[1,4]

$$\psi_{1,2}(t,x)=\pm\left[\frac{m}{2g^{2}}\gamma(1\pm v)\right]^{1/2}\sin\theta\times\frac{\exp[im\gamma\cos\theta(t-vx)+i\phi]}{\cosh[m\gamma\sin\theta(x-q-vt)\pm i\theta/2]}. \tag{7}$$

Furthermore, from the functional form of (5) and (7) and their covariance properties under the action of the Poincaré group it follows at once that the relativistic transformation laws of (q,v,θ,ϕ) are the same in both models. Thus, from the kinematical point of view the solitons arising in the MT and CSG models represent the same relativistic dynamical system. The Hamiltonian analysis[1] of this system shows that it may be interpreted as a particle with a pulsating internal degree of freedom which is described by the variable ϕ and whose period of motion in the center-of-mass frame is $2\pi/m\cos\theta$. Its corresponding mass, which can be calculated from either (5) or (7), turns out to be[3,4]

$$M=\frac{2m}{g^{2}}\sin\theta. \tag{8}$$

In spite of their kinematical equivalence there is, however, a difference between the solitons of the CSG and MT models: they have different charges given by $Q_T=(2/g^2)\theta$ for the MT model[4] and

$$\begin{aligned}&Q_{\mathrm{CSG}}=\frac{2}{g^{2}}\theta,\quad 0<\theta<\frac{\pi}{2},\\&Q_{\mathrm{CSG}}=0,\quad \theta=\frac{\pi}{2},\\&Q_{\mathrm{CSG}}=\frac{2}{g^{2}}(\theta-\pi),\quad \frac{\pi}{2}<\theta<\pi\end{aligned} \tag{9}$$

for the CSG model.[5]

In order to consider the classical S matrix for the CSG model it is convenient to introduce the scattering-data variables[3] $(k_0,c_0,a_0(k))$ of a pure one-soliton solution of the form (5). These variables are related to the parameters (q,v,θ,ϕ) in the form[6]

$$k_{0}=\tfrac{1}{2}\exp(-\beta+i\theta),\quad a_{0}(k)=\exp\left[-i\frac{g^{2}}{2}Q_{\mathrm{CSG}}\right]\frac{k-k_{0}}{k-k_{0}^{*}}, \tag{10a}$$

$$q=(m\sin\theta\cosh\beta)^{-1}\ln|c_{0}|,\quad v=\tanh\beta,\quad \phi=\arg c_{0}. \tag{10b}$$

Let us consider now the collision of two solitons with scattering data $(k_i, c_i^{\pm}, a_i(k))$ $(i=1,2)$ as $t \to \pm\infty$. By using the standard procedure[4,7] it is easy to deduce that provided $\beta_1 > \beta_2$ we have

$$\frac{c_1^+}{c_1^-} = a_2(k_1)^{-2}, \quad \frac{c_2^+}{c_1^-} = a_1(k_2)^2 \ . \tag{11}$$

Therefore, taking into account Eqs. (10a) and (10b) it follows that the soliton shifts are

$$\Delta q_i = -\epsilon(\beta_i - \beta_j) 2(m \sin\theta_i \cosh\beta_i)^{-1} \times \ln|S(\beta_i - \beta_j, \theta_i, \theta_j)| \ , \tag{12a}$$

$$\Delta\phi_i = \epsilon(\beta_i - \beta_j) 2[\theta_i - \arg S(\beta_i - \beta_j, \theta_i, \theta_j)] \ , \tag{12b}$$

where $\epsilon(\beta_i - \beta_j)$ is the sign of the relative rapidity and

$$S(\beta, \theta_1, \theta_2) = \frac{1 - \exp[\beta - i(\theta_1 - \theta_2)]}{1 - \exp[\beta - i(\theta_1 + \theta_2)]} \ . \tag{13}$$

These expressions coincide exactly with the soliton shift formulas corresponding to the MT model.[1,4] Thus, we see that a deep analogy exists between the soliton dynamics of the CSG and MT models. On the other hand, we may now use the results[1] on the SG-MT connection to reproduce the soliton dynamics of the SG model in terms of the CSG model. In this way, one finds the following identifications: first, states of the CSG soliton with $\theta = \pi/2$ represent SG kinks; secondly, states of a cluster consisting of two CSG solitons with parameters $(q_i, v_i, \theta_i, \phi_i)$ $(i=1,2)$ such that $q_1 = q_2$, $v_1 = v_2$, $\theta_1 + \theta_2 = \pi$ $(0 < \theta_1 < \pi/2)$, $\phi_1 + \phi_2 = 0$, represent SG breathers with parameters $(q, v, \theta, \phi) = (q_1, v_1, \theta_1, 2\phi_1)$. Through this correspondence and using Eq. (12) one recovers the soliton shift formulas for the SG model. It must be noticed that in this description of SG solitons only states of the charge-zero sector of the CSG model are used.

This work was partially supported by the Comisión Asesora de Investigación Científica y Técnica. We thank Professor H. de Vega for pointing out to us the correct expression (9) for the charge of the MT soliton.

[1]L. Martínez Alonso, Phys. Rev. D **30**, 2595 (1984).

[2]K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976); F. Lund and T. Regge, Phys. Rev. D **14**, 1524 (1976); F. Lund, Phys. Rev. Lett. **39**, 1175 (1977); B. S. Getmanov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 132 (1977) [JETP Lett. **25**, 119 (1977)]; F. Lund, Ann. Phys. (N.Y.) **115**, 251 (1978).

[3]H. J. de Vega and J. M. Maillet, Phys. Rev. D **28**, 1441 (1983).

[4]E. A. Kuznetsov and A. V. Mikhailov, Teor. Mat. Fiz. **30**, 303 (1977) [Theor. Math. Phys. (USSR) **30**, 193 (1977)].

[5]The expression of the charge Q_{CSG} of the CSG field is obtained from the Noether theorem and is given by

$$\int_{-\infty}^{\infty} \frac{i}{2(1-g^2|\psi|^2)} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) dx \ .$$

We notice that in Eq. (2.45) of Ref. 3 the value $(2/g^2)\theta \times \mathrm{sign}(\pi/2 - \theta)$ for the soliton charge is stated; however a careful integration shows that the correct value is the one given in Eq. (9).

[6]We adopt the same scattering-data variables as in Eqs. (2.8)–(2.15) of Ref. 3. It must be realized that the functional $\mu(\infty)$ appearing in Ref. 3 is equal to $-(g^2/2)Q_{\mathrm{CSG}}$.

[7]S. V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys. JETP **40**, 269 (1975)].